Silicon Photonics Laboratory
Introduction for Student's Guidance

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Literature

[Sadiku, 2009] Matthew Sadiku.
Numerical Techniques in Electromagnetics, Third Edition,
ISBN 978-1-4200-6309-7, 2009.

[Tavlove, 2005] Allen Tavlove and Susan Hagness.
Computational Electrodynamics,
The Finite Difference Time Domain Method,
ISBN-9-781580-538329, 2005.

Classification of electromagnetic problems

Cathegorization of problems

To answer the question of what method is best for solving a problem requires a classification of problems.

Problems are categorized by the

- solution region R
- equations describing the problem
- boundary conditions on S

These items define a problem uniquely.

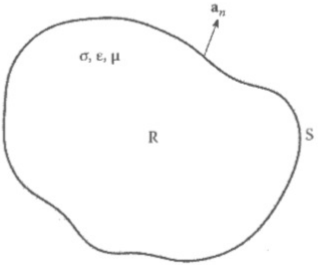


Figure: Solution region R with boundary S. (Sadiku 2009, p.15)

Finite Differencing

When should finite differences be applied?

It is very rarely in real applications that an analytic solution can be found. The analytic approach fails if

- the PDE is not linear and cannot be linearized without affecting the result
- the PDE is supposed to be solved in a complex region
- the boundary conditions are of mixed type
- the boundary conditions are time-dependent
- the medium is inhomogeneous or anisotropic

Problems with such complexity require numerical methods.

The methods of finite differences is one of the most frequently used and more generally applicable than any other method.

Common grid patterns used in finite difference methods

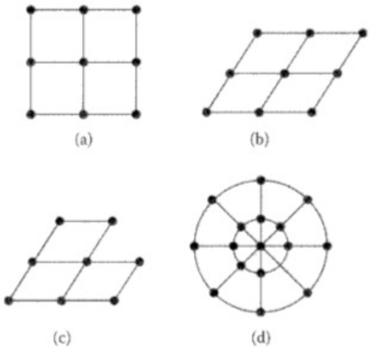


Figure: Grid patterns. a) rectangular grid, b) sjew grid, c) triangular grid, d) circular grid. (Sadiku p.120)

Finite difference of a first order approximation

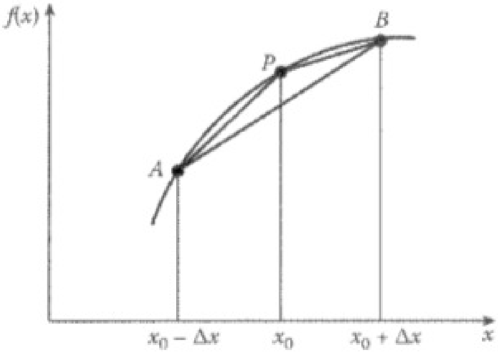


Figure: Estimates for the derivative of $f(x)$ and P using forward, backward, and central differences (Sadiku p.121)

First order center difference approximation

The center difference approximation for the first order derivative of f and a dependent variable x at a point P is obtained from

$$f'(x) \approx \frac{f(x_0 + \triangle x) - f(x_0 - \triangle x)}{2\triangle x}$$

where the central-difference is obtained from the arc AB.

Second order center difference approximation

The center difference approximation for the second order derivative of f and a dependent variable x at a point P is obtained from

$$f''(x) \approx \frac{f(x_0 + \triangle x) - 2f(x_0) + f(x_0 - \triangle x)}{(\triangle x)^2}$$

where the center difference is obtained from the arc AB.

Classification of Systems

Linearity

The **linearity assumption** holds for all systems when then superposition property is obeyed for all input functions p and q as well as all complex constants a and b. Assuming a, for the moment not necessarily defined, operator or system $\mathcal{S}$, the system is said to be linear if

$$\mathcal{S}(ap + bq) \quad = \quad a \cdot \mathcal{S}(p) + b \cdot \mathcal{S}(q)$$

Example-1: Is $f(x) = 3x$ a linear function? Yes,

$f(ap + bq) = 3(ap + bq) = a(3p) + b(3q) = af(p) + bf(q).$

Example-2: Is $f(x, y) = 3x + 2y$ a linear function? Yes,
$f(ap+bq, ap+bq) = 3(ap+bq)+2(ap+bq) = af(p, q)+bf(p, q)$

Example-3: Show that $f(x) = x^2$ is not a linear function.

Invariance

A system $\mathcal{S}$ is called **invariant** if the response of $\mathcal{S}$ at a time t on an excitation applied at time t' only depends on the time difference $t - t'$. Electrical networks composed of fix resistors, capacitors and inductors are time-invariant as the properties of the components (ideally) do not change over time.

Similar to the time-invariance, an optical system is said to be **space-in-variant** (isoplanatic), if the response of the system $\mathcal{S}$ at a location $\mathbf{r}'$ on an excitation at a location $\mathbf{r}$ only depends on the distance $\mathbf{r} - \mathbf{r}'$. For an invariant system, the superposition integral takes on the form of the **convolution integral**, where $u(x, y)$ is the excitation and $h(x, y)$ the **spatial impulse response**.

$$u'(x, y) \;\; = \;\; \int\int_{-\infty}^{\infty} u(x', y') \cdot h(x - x', y - y') \; dx' \; dy'$$

Coherency

Incoherent systems are like AM radio (amplitude modulation). The information is encoded in the amplitude (e.g. turning the light on/off) and the receiver, e.g. a photo diode, has to measure the intensity to decide upon the encoded information.

Coherent systems are like FM radio (frequency modulation). The information is encoded in the property of the signal, i.e. phase, amplitude and frequency of the field. The receiver is much more complicated and the signal has to provide some coherency, i.e. phase and frequency relation.

In this sense, a monochromatic wave, i.e. a wave of one frequency, is perfectly coherent. Sunlight or the light coming from a light bulb, which evolves from uncorrelated oscillators, are uncorrelated sources.

Introduction to Fourier Optics

The Dirac delta function

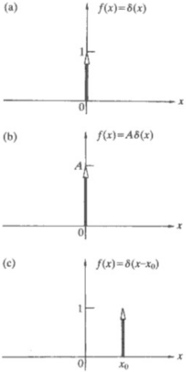


Figure: Hecht p.527

$$\delta(x-x_0) = \begin{cases} 0 & x \neq x_0 \\ \infty & x = x_0 \end{cases}$$

$$A \cdot \int_{-\infty}^{\infty} \delta(x)\, dx = A \cdot 1$$

$$\int_{-\infty}^{\infty} f(x) \cdot \delta(x-0)\, dx = f(0)$$

$$\int_{-\infty}^{\infty} f(x) \cdot \delta(x-x_0)\, dx = f(x_0)$$

One-Dimensional Fourier Transformation

According to the Fourier theorem, one-dimensional periodic function $f(x)$ can be expressed as a linear combination of harmonic functions

$$f(x) = \frac{1}{\pi}\left[\int_0^\infty A(k)\cos kx \; dk + \int_0^\infty B(k)\sin kx \; dk\right]$$

where k are the angular spatial frequencies. The factors

$$A(k) = \int_{-\infty}^{\infty} f(x)\cos kx \; dx \qquad B(k) = \int_{-\infty}^{\infty} f(x)\sin kx \; dx$$

determine the weights for the respective sine and cosine transforms. In a complex representation $F(k) = A(k) + jB(k)$.

Spatial frequency and propagation vector

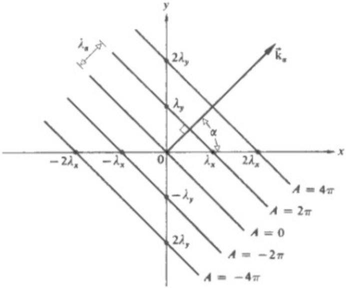


Figure: (Source: Optics, E. Hecht, p.522)

$$\alpha = \arctan \frac{k_y}{k_x} = \arctan \frac{\lambda_y}{\lambda_x}$$

$$\lambda_\alpha = \frac{1}{\sqrt{\lambda_x^{-2} + \lambda_y^{-2}}}$$

$$k_\alpha = \sqrt{k_x^2 + k_y^2}$$

$$k_x = \frac{2\pi}{\lambda_x} = 2\pi\nu_x \quad k_y = \frac{2\pi}{\lambda_y} = 2\pi\nu_y$$

$$k_z = \frac{2\pi}{\lambda_z} = 2\pi\sqrt{\frac{1}{\lambda^2} - \frac{1}{\lambda_x^2} - \frac{1}{\lambda_y^2}}$$

$$k_\alpha = \sqrt{k_x^2 + k_y^2}$$

Finite difference approximation of parabolic PDEs

The approximation is only stable and accurate for $0 < r \leq 1/2$. For $r = 1/2$, the explicit equation on page 7 simplifies to

$$\Phi(i, j+1) \;\; = \;\; \frac{1}{2}\left[\Phi(i+1, j) + \Phi(i-1, j)\right]$$

and the computational molecule is shown in the right figure on page 8.

An **implicit formula**, proposed by Crank and Nicholson in 1974 is valid for all finite values of r, where the second order space derivative in x is replaced by the average of the center difference formulas on the j-th and $(j+1)$-th time rows.

Crank-Nicholson Algorithm (Crank and Nicholson, 1974)

The implicit formula derives three unknown values of Φ (circles) from three known values of Φ (rectangles). For nodes $j = 0$ and $i = 1, 2, ..., n$ This results in n equations for n unknowns and similar for $j = 1$ and so forth. The method allows a much larger time step $\triangle t$ and is stable for all finite r.

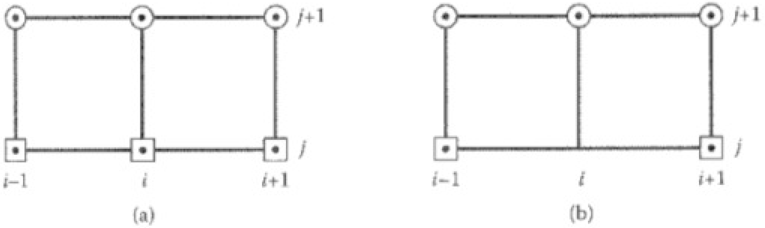


Figure: Computational molecule for Crank Nicholson method: a) for finite values or r, b) for $r = 1$. (Sadiku p.125)

Wave propagation

$$\boxed{\Phi_{xx} = \Phi_{tt}} \quad (\textit{Wave Equation})$$

Solve the wave equation in the region $0 < x < 1$ for $t \geq 0$, considering the boundary conditions

$$\boxed{\Phi(0,t) = \Phi(1,t) = 0}$$

and the initial conditions

$$\boxed{\Phi(0,t) = \sin \pi x} \quad \boxed{\Phi_t(x,0) = 0}$$

.

(Remarks: $\Phi_{tt} = \frac{\partial^2 \Phi}{\partial t^2}$, $\Phi_{xx} = \frac{\partial^2 \Phi}{\partial x^2}$ and $\Phi_t = \frac{\partial \Phi}{\partial t}$)

Finite difference approximation of the wave equation

Using the explicit finite difference scheme introduced on page 16 and with $r = 1$, the finite difference equation is

$$\Phi(i, j+1) \;=\; \Phi(i-1, j) + \Phi(i+1, j) - \Phi(i, j-1) \;, \quad j \geq 1$$

For $j = 0$, the boundary condition yields

$$\Phi_t \;=\; \frac{\Phi(i,1) \;-\; \Phi(i,-1)}{2\triangle t} \;=\; 0$$

which gives the expression $\Phi(i,1) = \Phi(i,-1)$. Inserting this condition into the finite difference equation and with $v = r = 1$ and $\triangle t = \triangle x$

$$\Phi(i,1) \;=\; \frac{1}{2}\;[\;\Phi(i-1,0) \;+\; \Phi(i+1,0)\;]$$

Analytic solution of the wave equation

Without further explanation and analysis, the analytic solution of the wave equation is

$$\Phi(x, t) \;\; = \;\; \sin(\pi x) \cdot \cos(\pi t)$$

This solution can be used to verify the accuracy of the finite difference method.

Implementation

Implement the algorithm, e.g. Matlab, C/C++, Java ...
and verify the accuracy against the analytic solution.

Roundoff and discretization errors

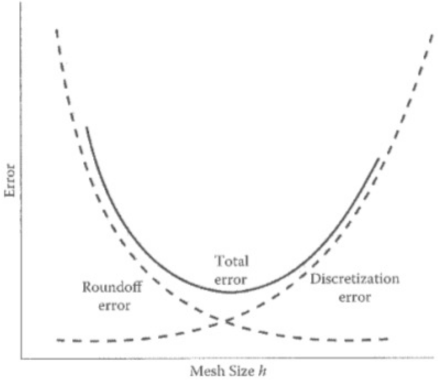


Figure: Error as a function of the mesh size. (Sadiku p.143)

Finite difference approximation of Maxwell's curl equations

The Finite Difference Time Domain (FDTD) method is a convenient tool for wave scattering problems. The FDTD method has been introduced by Yee in 1966. It was then developed by Tavlove and others.

The method evolved from a direct/rigorous solution of the Maxwell equations using finite difference schemes in an interleaved half-step solution with respect to space and time and as an initial value problem. The central equations are Maxwell's curl eq-uations, where the interactions of the electric and magnetic field components in an isotropic medium are

$$\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t} \qquad \nabla \times \mathbf{H} = \sigma \mathbf{E} - \epsilon \frac{\partial \mathbf{E}}{\partial t}$$

Yee's unit cell

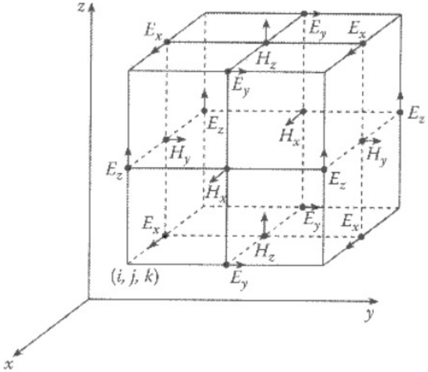
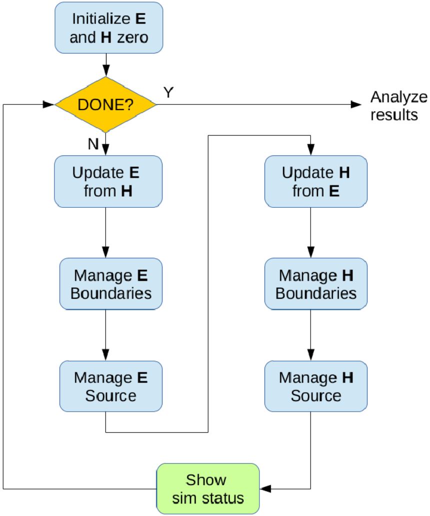


Figure: Positions of the field components in a unit cell of the Yee lattice. (Source: Numerical Techniques in Electromagnetics, p.161, Sadiku 2009)

FDTD Flow


Initialize **E** and **H** zero
DONE?
Y
Analyze results
N
Update **E** from **H**
Manage **E** Boundaries
Manage **E** Source
Update **H** from **E**
Manage **H** Boundaries
Manage **H** Source
Show sim status

The Helmholtz equation with plane waves

Using plane waves of the form $u(\mathbf{r}) = U_0 \cdot e^{j\mathbf{k}\cdot\mathbf{r}-\omega\cdot t}$ in the partial differential wave equation yields the **Helmholtz equation**

$$\left(\nabla + k^2\right) \cdot u(\mathbf{r}) \quad = \quad 0$$

with $k^2 = (n \cdot \omega / c)^2$. From transversality and $|k|^2 = k_x^2 + k_y^2 + k_z^2$ the vector components of $\mathbf{k}$ are dependent. This yields the expression for k_z

$$k_z \;=\; \pm\sqrt{\left(\frac{n\cdot\omega}{c}\right)^2 - k_\perp^2} = \pm\sqrt{\left(n\cdot k_0\right)^2 - k_\perp^2} = \pm 2\pi\sqrt{\left(n\cdot\nu_0\right)^2 - \nu_\perp^2}$$

with $\nu_\perp^2 = \nu_x^2 + \nu_y^2$. Signs indicate a propagation in the positive or negative direction of the propagation vector. At boundaries, $k_\perp$ is maintained and k_z changes with $\triangle n$.

Transformation of spatial frequencies at a boundary

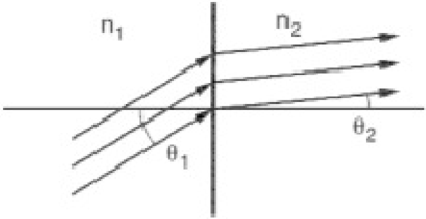


Figure: Goodman p.32

Propagating and evanescent plane waves

The solution for k_z provides two different type of solutions, whereof one is connected to propagating and one to so called evanescent waves. The two soltions of the square root are

$$|\mathbf{k}_\perp| \leq \frac{n\omega}{c} \quad \textit{propagating waves}$$
$$|\mathbf{k}_\perp| > \frac{n\omega}{c} \quad \textit{evanescent waves}$$

where in case of $|\mathbf{k}_\perp| = n \cdot \omega / c$, the wave propagates perpendicular to the z-axis, the selected axis of propagation according to the propagator equation on page 11. In case of evanescent waves, $\mathcal{P}$ becomes real-valued and for a distance $\triangle z$ equal to

$$\mathcal{P}_{ev} = e^{-\triangle z \cdot \sqrt{k_\perp^2 - \left(\frac{n\omega}{c}\right)^2}}$$

Evanescent waves

In case of $|\mathbf{k}_\perp| > n \cdot \omega / c$, the propagator becomes a real-valued expo- nential function, which decays exponentially along the z-axis.

$$\mathcal{P}_{ev}(\mathbf{k}_\perp, z) \quad = \quad e^{j \cdot z \cdot \sqrt{k_\perp^2 - \left(\frac{n\omega}{c}\right)^2}}$$

The **entire plane evanescent wave** is then composed from the evanes- cent propagator $\mathcal{P}_{ev}$ multiplied by the perpendicular wave component, as defined by the spatial frequency $\mathbf{k}_\perp = 2\pi\nu_\perp$

$$u_{ev}(\mathbf{r}_\perp, z) \quad = \quad e^{j \cdot \mathbf{k}_\perp \cdot \mathbf{r}_\perp} \cdot e^{j \cdot z \cdot \sqrt{k_\perp^2 - \left(\frac{n\omega}{c}\right)^2}}$$

where $|2\pi\nu_\perp| > \frac{n\omega}{c}$. The **rate of decay** depends on the square root and the absolute reflect that there are **positive and negative evanescent frequencies**.

Expansion of plane wave components

Inserting the propagator $\mathcal{P}(\mathbf{k}_\perp, z)$ into equation 1 and 1, the propa- gation of plane wave components along the z-axis over a distance z is obtained from the inverse Fourier transformation

$$u(\mathbf{r}_\perp, z) \;=\; \mathcal{F}^{-1}\left\{U(\mathbf{k}_\perp)\cdot\mathcal{P}(\mathbf{k}_\perp, z)\right\}$$

where $U(\mathbf{k}_\perp) = \mathcal{F}\left\{u(\mathbf{r}_\perp, 0)\right\}$. In case of **homogeneous media**, the calcu- lation provides the exact result, called the **plane wave decomposition**. According to the **convolution theorem**, the corresponding expression in the space domain is

$$u(\mathbf{r}_\perp, z) \;=\; u(\mathbf{r}_\perp, 0) * \mathcal{F}^{-1}\left\{\mathcal{P}(\mathbf{k}_\perp, z)\right\}$$

This is the convolution kernel in the space domain for propagat- ion $\mathcal{F}^{-1}\left\{\mathcal{P}(\mathbf{k}_\perp, z)\right\}$.

Sampling Theorem and Nyquist frequency

The sampling or **Nyquist theorem** states that a signal with a maximum frequency ν_{max} in its spectrum needs to be sampled with a sampling frequency $\nu_s > 2 \cdot \nu_{max}$.

For a given sampling freqency ν_s, the **Nyquist frequency** $\nu_N = \nu_s/2$ gives the highest eligible frequency in the spectrum to reconstruct signals without aliasing, i.e. overlap of base- and high-order bands.

The **Nyquist condition** is

$$\nu \quad < \quad \nu_N$$

The spectrum of a signal has to be band-limited to the Nyquist frequency to avoid distortions in the spatial domain from aliasing. Such band limitation filters are called anti-aliasing filters.

Split Step Propagation Scheme

The basic concept of the **Split Step Propagation Scheme** is the separa- tion of the refractive index into an average component $\overline{n}$ and a variation from the average $\triangle n$.

With this scheme, a propagation through homogeneous medium $\overline{n}$ in the spatial frequency domain is combined with the phase correct- ion from deviations to the average $\triangle n$ in the spatial domain.

A **fundamental problem** arising from this concept is caused by the situation that plane wave components are only available in the frequency or Fourier domain and the corresponding plane wave components propagate in a medium with average refractive index while the phase correction needs to consider the spatial distribution of the refractive index and thus needs to be performed in the space domain where no spatial frequencies are available.

Propagation in inhomogeneous medium

In case of **inhomogeneous media**, a propagating wave traverses regions of different refractive index, $n(z) \neq n(z + \triangle z)$.

The **Wave Propagation Method** (WPM) derives the field distribution of a propagating wave in a two- or three-dimensional system from a plane wave decomposition of the incident wave, Snell's law and the Fresnel amplitude coefficients, which can be reformulated for spatial frequencies.

Inhomogeneities in the refractive indes cause a transformation in the z-component of the spatial frequency vector and a transformation of amplitudes. The Wave Propagation Method does not apply a split-step propagation scheme as for example the Beam Propagation Method.

Wave Propagation Scheme

The fundamental limitation of all flavours of the Beam Propagation Method, even with extensions to wide angle propagation is completely eliminated by the Wave Propagation Scheme.

The **Wave Propagation Scheme** (Brenner and Singer, 1993) derives the field distribution of a field propagated through inhomogeneous medium from a superposition of plane waves, considering the spatial distribution of the refractive index without approximation like in the Split Step Scheme.

The **first step** of the Wave Propagation Scheme is identical to the Split Step Scheme, the Fourier transformation of the incident field distribution.

$$U(\mathbf{k}_\perp) = \mathcal{F}\left\{u(\mathbf{r}_\perp)\right\}$$

where the Fourier coefficients U correspond to amplitudes of plane waves propagating at the corresponding spatial frequency.

Wave Propagation Scheme

All plane wave components in the spectrum of the incident field propagate through the refractive index distribution $n(\mathbf{r}_\perp)$ and the field at a location $\mathbf{r}_\perp$ at $z = \mathtt{i} \cdot \mathtt{dz} + \mathtt{dz} = (\mathtt{i} + 1) \cdot \mathtt{dz}$ after a propagation distance $\mathtt{dz} = \mathtt{Z}/\mathtt{nz}$ is obtained from a field distribution at $z = i \cdot \triangle z = \mathtt{i} \cdot \mathtt{dz}$

$$u(\mathbf{r}_\perp, z + \triangle z) \;=\; \frac{1}{A} \iint\limits_{k_\perp} U(\mathbf{k}_\perp, z) \cdot \mathcal{P}(\mathbf{k}_\perp, \mathbf{r}_\perp, \triangle z)\, d\mathbf{k}_\perp$$

where $A = X \cdot Y$ and with the Wave Propagation Scheme, the propagator $\mathcal{P}$ depends on the spatial frequency $\mathbf{k}_\perp$ and space $\mathbf{r}_\perp$.

$$\mathcal{P}(\mathbf{k}_\perp, \mathbf{r}_\perp, \triangle z) \;=\; e^{\,j \triangle z \cdot \sqrt{(n(\mathbf{r}_\perp) \cdot k_0)^2 - k_\perp^2}}$$

Vector Wave Propagation

Scalar methods cannot simulate vectorial characteristics of electromagnetic waves. This is a significant limitation for wide angles as the amplitudes of TE- (S-) and TM- (P-) polarized waves show characteristic behaviour in case of inhomogeneities as described by the Fresnel amplitude coefficients.

The **Vector Wave Propagation Method** extends the Wave Propagation Method by the full vector geometry at interfaces. The amplitudes of TE- and TM-polarized fractions of waves are scaled by the Fresnel coefficients of transmission in the one-directional case.

The bi-directional Vector Wave Propagation Method follows a special scheme to consider transmitted and reflected vector waves from vector geometry and Fresnel amplitude coefficients of reflection.

Spectrum of vector waves

The propagating wave in the Vector Wave Propagation Scheme is a vector

$$\mathbf{U}(\mathbf{k}_\perp) \;=\; \begin{pmatrix} U_x(\mathbf{k}_\perp) \\ U_y(\mathbf{k}_\perp) \\ U_z(\mathbf{k}_\perp) \end{pmatrix}$$

obtained from two-dimensional Fourier transformations of each vector component in the tree-dimensional spatial field distribution vector $\mathbf{u}(\mathbf{r}_\perp)$

$$\mathbf{U}(\mathbf{k}_\perp) \;=\; \frac{1}{A}\iint\limits_{A} \mathbf{u}(\mathbf{r}_\perp) \cdot e^{-j\cdot\mathbf{k}\cdot\mathbf{r}}\, d\mathbf{r}_\perp$$

Propagation of vector waves

A plane wave component $\mathbf{W}$ propagating a distance $\triangle z$ is then obtained from a plane wave expansion of $\mathbf{U}$ over the aperture A from

$$\mathbf{W}(\mathbf{k}_\perp, \mathbf{r}_\perp, \triangle z) \;=\; \mathbf{U}(\mathbf{k}_\perp) \cdot e^{\,j\mathbf{k}_\perp \cdot \mathbf{r}_\perp + k_z \cdot \triangle z}$$

where k_z is a space and frequency dependent quantity as already introduced in the Wave Propagation Method.

In case of a homogeneous medium, no boundary affects the direction of $\mathbf{k}$ and $\mathbf{W}$ is only affected by a phase shift and the plane wave decomposition is applicable. In case of an inhomogeneities, the vector geometry at boundaries is relevant, where $\mathbf{U}$ is split into a reflected vector $\mathbf{U}_r$ and a transmitted vector $\mathbf{U}_t$ and into a TE-/S- and TM-/P- mode as demanded by laws of reflection and refraction.

Implementation (2/2)

The spatial distribution is then obtained from four sums

$$\mathtt{u_{i+1}} \;=\; \frac{1}{\mathtt{X} \cdot \mathtt{Y}} \cdot \sum_{\mathtt{p}} \sum_{\mathtt{q}} \sum_{\mathtt{l}} \sum_{\mathtt{m}} \mathtt{T_{i,i+1}} \cdot \mathtt{U_i} \cdot \mathtt{P_{i+1}}$$

$$\mathtt{T_{t,i,i+1}(p,q,l,m)} \;=\; \frac{1}{\mathtt{k_\perp^2}} \begin{bmatrix} k_y^2 t_{te} + k_x^2 t_{tm} & k_x k_y (t_{tm} - t_{te}) \\ k_x k_y (t_{tm} - t_{te}) & k_x^2 t_{te} + k_y^2 t_{tm} \end{bmatrix}$$

$$\mathtt{U_i(p,q)} \;=\; \mathtt{FFT2D(\ u_i(l,m)\ )}$$

$$\mathtt{P_{i+1}(p,q,l,m)} \;=\; \mathtt{e}^{\,\mathtt{j \cdot dz \cdot} \sqrt{\mathtt{n_{i+1}^2(l,m) \cdot k_0^2} \;-\; \mathtt{((p/X)^2 + (q/X)^2)}}}$$

where $\mathtt{p, l} \in \{-\mathtt{nx}/2, .., \mathtt{nx}/2-1\}$, $\mathtt{q, m} \in \{-\mathtt{ny}/2, .., \mathtt{ny}/2-1\}$ and $\mathtt{i} \in \{0, .., \mathtt{nz}-1\}$.